Arithmetic mean: a bellwether for unbiased forecasting of portfolio performance

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Abstract

Purpose – We know that estimates of terminal value of long-term investment horizons are biased. Unbiased estimates exist only for investment horizon of one time-period. The purpose of this paper is to suggest a method based on the arithmetic mean in order to obtain unbiased estimates for the terminal value of long-term investment horizons.

Design/methodology/approach – The method used for the investigation was to employ loss functions or error statistics. Namely, the mean error, the mean absolute error, the root mean squared error, and the mean absolute percentage error was used.

Findings – The suggested method produced the closest values to the actual ones than any other suggested averaging method when the authors examined ten-year investment horizons for Standard & Poor's 500 index and on Dow Jones Industrial index.

Practical implications – Portfolio managers and individual investors may use this paper's suggestion if they wish to obtain unbiased estimates for investment horizons greater than one time-period.

Originality/value – The suggestion to equate the time-period of the observed data to the time-period of the investment horizons is novel and useful to practitioners since it produces unbiased estimates.

Keywords Portfolio investment, Estimation, Long-term planning, Arithmetic, Geometric mean

Paper type Research paper

1. Introduction

In financial management, performance evaluation is a crucial aspect. Financial managers often face the question "what wealth a portfolio is expected to generate over a long-term?" In order to obtain an answer suitable for managerial decisions, they must produce estimates of expected long-term returns. Expected returns are usually estimated by averaging past returns. A procedure which is logical, simple, and in many cases, is appropriate. Average returns are affected by the start and terminal periods used in measurement. Among financial professionals, the most common mean returns used are the arithmetic and the geometric ones. Given measures of returns on a financial asset, or index of returns on a portfolio of such asset, over a series of time-periods, the arithmetic mean is the simple average of percentage returns for each time-period and the geometric mean is calculated from the index of returns and reflects the compound rate of return over the investment horizon.

It is well known, however, that both these mean returns produce biased estimates when the investment horizon is greater than the unit-time (Indro and Lee, 1997). The

Managerial Finance
Vol. 36 No. 11, 2010
pp. 958-968

0307-4358
DOI 10.1108/03074351011081277

arithmetic mean return of past returns will overestimate the expected terminal return, whereas the geometric will underestimate it (Blume, 1974; Indro and Lee, 1997; Mayo, 2006). The magnitude of the bias grows with the length of the investment horizon. The choice between which mean to use is a mater of academic debate among researchers. There has been a plethora of papers about the relative merits of using the arithmetic mean vs the geometric mean. The arithmetic mean receives the most support in the literature, other recommend the geometric, and a few support something in between such as a weighted average of the two. See Fabozzi (1999) and Francis and Ibbotson (2002) to quote a few.

In this paper we address the problem of bias and we investigate the use of arithmetic mean in within different time-periods in order to produce unbiased estimates for investment horizons greater than one time-period.

The rest of the paper is organized as follows: We first discuss the use of arithmetic and geometric means in portfolio performance as well as some mixtures of both. Next, we present the bias problem and our suggestion for overcoming it. Then, we demonstrate the data employed in our analysis. After that, we provide the empirical comparisons of five averaging measures (arithmetic, geometric and harmonic means, and Blume's and Cooper's weighted suggestions) with respect to four error statistics (mean error (ME), mean absolute error (MAE), root mean squared error (RMSE) and mean absolute percentage error (MAPE)). Finally we conclude.

2. Arithmetic mean vs geometric mean

When estimating the average rates of change based on historical data, the vast majority of the analysts and/or managers favor two approaches in order to calculate the historical mean: the arithmetic mean and the geometric mean. Both means assume a standard and constant method of transforming the initial value of given historical data to the terminal value of a given investment horizon. Given annual data, the arithmetic mean assumes that each year's return is a constant number (the arithmetic average of historical data). Similarly, the geometric mean assumes that a constant annualized measure of the proportional change in value occurs until the terminal value. In other words, it assumes that the initial value grew at a constant rate of return.

It is well known that the mathematics of geometric mean require positive numbers. On the other hand, financial rate of returns could be any number, positive or negative. Therefore, a transformation is required in order to make our data applicable to all means. We transform each rate of return to an index of return, i.e. if R_t is the rate of return for year t, we compute the index r_t as $r_t = 100 + R_t$ and we base all our computations on r_t.

Mathematically speaking, the arithmetic mean is defined as the sum of the values of a random variable divided by the number of values. If r_t, $t = 1, \ldots, T$ denote the T observed values of random annual returns (index form), then the arithmetic mean of r, a(r), is:

$$a(r) = \frac{1}{T}\sum_{t=1}^{T} r_t. \tag{1}$$

Accordingly, the geometric mean of a set of T positive values is defined as the Tth root of the product of all the values of the set. The geometric mean of r_t, g(r), is:

$$g(r) = \sqrt[T]{r_1 r_2 \ldots r_T}. \tag{2}$$

A second and more practical way to calculate the geometric mean is to take the antilog of the arithmetic average of the logarithms of r's, i.e.:

$$g(r) = \text{anti}\log\left(\frac{1}{T}\sum_{t=1}^{T}\log(r_t)\right) = e^{\frac{1}{T}\sum_{t=1}^{T}\log(r_t)}. \quad (3)$$

As Pythagoras already proved, the geometric mean is always less than or equal to the arithmetic mean. Approximately, the following formula holds:

$$a(r) \cong g(r) + \frac{1}{2}\text{Var}(r). \quad (4)$$

This can be interpreted the following way: The exact relationship between the two means will depend on volatility and time-period measured. The larger the volatility, the larger the difference between the two means. Equality holds if and only if all members of the data set are equal, i.e. when $r_1 = r_2 = \ldots = r_T$.

Assuming that the past is indicative of the future, the arithmetic mean is a better measure of expected future return. Geometric mean is a better measure of past performance over some specified period of time (Francis and Ibbotson, 2002).

The difference between the two approaches basically arises due to the variance of returns. Arithmetic returns respond to volatility, and do not measure the actual return by investors over a period of more than one year. Hence the more variable returns have been, the more arithmetic means will depart from actual returns. When earning a constant annual return, the geometric mean will be the same as the arithmetic mean.

The difference between arithmetic and geometric mean returns can be considerable. Ibbotson (2002), examining returns in US capital markets over the period 1920-2001, reports that the arithmetic mean large company stocks return is 18.69 percent greater than its geometric counterpart. The corresponding figure for bonds was 29.63 percent greater.

Geometric mean return should be used for measuring historical returns that are compounded over multiple time-periods. Arithmetic mean return should be used for future-oriented analysis where the use of expected values is appropriate. Deciding which measure of mean tendency to report will depend on the type of data collected and the distribution of the results. The geometric mean is not as sensitive to extreme figures and skewness as the arithmetic is. In general, the arithmetic mean is appropriate when values follow a more-or-less normal distribution with no outliers.

Keeping in mind that we are dealing with returns, one more question arises. Time-series averaging should be arithmetic or geometric? Supporters of the latter view such as Copeland *et al.* (2000) and Damodaran (1999) are minority.

Arithmetic mean is unbiased, easier to calculate, easier to understand, and scientifically more meaningful. But, it is an unbiased estimate of, only, one-period (e.g. one year) expected returns. Problems of bias arise when we are interested in expected returns for time-periods greater than one. Compounding estimates rather than true expected values adds upward bias to measures of expected long-term wealth.

As Fama and French (1999, p. 1943) state:

> [...] under certain conditions, the appropriate return concept is an expected one-period simple return. This issue is complicated, however, by the fact that the expected returns must be estimated and the estimates enter present value expressions in a nonlinear way. As a consequence, estimates of the cost of capital that produce unbiased estimates of present values are weighted averages of average simple and compound returns, with weights that depend on the maturity of the cash flow to be valued.

In financial literature, various ways to produce unbiased estimates have been suggested. The most classical are those of Blume (1974). Blume initially, and Cooper (1996) later, showed that the arithmetic mean of historical returns is a less biased estimate than the geometric mean. Cooper's reasoning further assumes that returns are serially independent. However, when returns are negatively serially correlated the arithmetic average is not necessarily superior as a forecast of long-term future returns. Historical evidence suggests that the stock market is mean-reverting (Poterba and Summers, 1988). That is, periods of high returns tend to be followed by periods of lower returns. In such a case the geometric mean is a better indication of long-term future prospects. Damodaran (1999) reaches to the conclusion that the arithmetic mean return is likely to over state the premium since his empirical studies indicate that returns on stocks are negatively correlated over time. The statistical nature of the problem is such that the choice of mean ultimately will depend on the predictability of returns over longer time-periods and the distribution of these returns. The more unpredictable the return, the better the case for using the arithmetic average.

3. Mixtures of arithmetic and geometric means

In mathematical and financial literature, there have been several suggestions of using mixtures of arithmetic mean and geometric mean. From mathematics we obtain the logarithmic mean and the identric mean. For a very brief discussion on these means see Vamanamurthy and Vuorinen (1994).

Using our notation and assuming that the two means, the arithmetic and the geometric, are positive and not equal, their logarithmic mean is defined as:

$$L[a(r), g(r)] = \frac{a(r) - g(r)}{\log[a(r)] - \log[g(r)]}, \tag{5}$$

similarly, their identric mean is:

$$I[a(r), g(r)] = e^{-1}\left(\frac{[a(r)]^{a(r)}}{[g(r)]^{g(r)}}\right)^{\frac{1}{a(r)-g(r)}}. \tag{6}$$

The following inequalities hold:

$$g(r) \leq L[a(r), g(r)] \leq I[a(r), g(r)] \leq a(r). \tag{7}$$

Blume (1974) examined the above-mentioned bias problem under the assumption that returns are independently and normally distributed. For an investment horizon of N time-periods, he found that the arithmetic mean overestimates the true compound rate of return and the geometric mean underestimates it. Blume's remedy was to suggest using both arithmetic and geometric means of historical data to form an unbiased estimator of the expected return. His proposed estimate, say b(r), was formulated by weighting the compounded arithmetic and geometric means with N in the following way:

$$b(r) = \sqrt[N]{da^N(r) + (1 - d)g^N(r)}, \tag{8}$$

where

$$d = \frac{T - N}{T - 1}. \tag{9}$$

Note that when $N = 1$ then $b(r) = a(r)$, whereas when $N = T$ we get $b(r) = g(r)$.

Cooper (1996) and Jacquier *et al.* (2003; 2005) examined the case where the returns are log-normally distributed, instead of Blume's normality assumption, and suggested using:

$$c(r) = g(r) + \frac{k}{2}\mathrm{Var}(r), \tag{10}$$

where

$$k = \frac{T - N}{T}. \tag{11}$$

Obviously, according to (4), when $N = 0$ we have $c(r) \cong a(r)$, and when $N = T$ we get $c(r) = g(r)$.

4. The bias problem

At this point, it is most appropriate to quote Jacquier's *et al.* (2003) abstract of their paper:

> An unbiased forecast of the terminal value of a portfolio requires compounding of its initial value at its arithmetic mean return for the length of the investment period. Compounding at the arithmetic average historical return, however, results in an upwardly biased forecast. This bias does not necessarily disappear even if the sample average return is itself an unbiased estimator of the true mean, the average is computed from a long data series, and returns are generated according to a stable distribution. In contrast, forecasts obtained by compounding at the geometric average will generally be biased downward. The biases are empirically significant. For investment horizons of 40 years, the difference in forecasts of cumulative performance can easily exceed a factor of 2. And the percentage difference in forecasts grows with the investment horizon, as well as with the imprecision in the estimate of the mean return. For typical investment horizons, the proper compounding rate is in between the arithmetic and geometric values.

Many scholars have dealt with the existence of bias of estimates based on either the arithmetic mean or the geometric one. To quote a few, see Blume (1974), Cooper (1996), Indro and Lee (1997), and Jacquier *et al.* (2003 and 2005). Blume (1974) explains that even if returns are independent and normally distributed there will still exist an upward (downward) bias based on variance (time) for the arithmetic (geometric) mean when used as long-run estimates of future returns When the number of future periods (N) is larger than the number of historical observations (T) geometric mean will be downward biased. When $N > 1$ arithmetic mean will be upwards biased. He therefore suggests that the weighted average of the arithmetic mean and geometric mean, given in (8), should be used in order to not overstate and understate the true return. Cooper (1996) focuses upon correction of estimation errors in US data from 1926 to 1992 by measuring real returns and adjusting to allow for serial correlation in returns. He suggested using (10). The unbiased estimates of discount rates which he tentatively arrives at are far closer to the arithmetic than the geometric mean.

All those who investigated the bias problem agree that, as Booth (2006) states it very accurately:

> [. . .] in all cases the arithmetic return is the best estimate of next period's rate of return while the geometric return is only one estimate of what would have been earned over this very long investment horizon: neither are appropriate for investment horizons in between these two extremes.

Since the only case in which we receive unbiased estimates, using annual data say, is when we apply the arithmetic mean in order make forecasts for an investment horizon of one year. Here, the relation of annual data and investment horizon of one year is very important. In fact, we obtain unbiased estimates when we have historical data of some time-period (one year in our case) and we forecast for an interval of only one time unit (one year in our case). That is to say, if for example, we had monthly observations, we would receive unbiased estimates only if we applied the arithmetic mean in order to obtain forecasts for the next month: monthly observations, then investment horizon of a month; daily observations, then investment horizon of a day. Similarly for any suitable unit of time such as quarters, hours, five-year periods, or anything other.

According to the above, there is one question that needs an answer. For the sake of our argument, let us suppose that we have annual observations on a portfolio's returns over a series of years and we wish to forecast the portfolio's value in five years time. We know, however, that the forecasts of a five-year period (a period greater than annual) based on annual data, no matter the averaging method, are biased. Our idea and suggestion is extremely simple. If in our example, we deliberately ignore four years out of each group of five years, we compute the arithmetic mean of the remaining observations, and we forecast the value of the portfolio five years ahead – we end up with an unbiased result. In other words, if our first priority is the unbiasness of our results, we only use the historical observations with time distance exactly equal to the investment horizon.

5. Empirical analysis

For the fairness of our results, we employed real life and easily accessible data on Standard & Poor's 500 (SP500) index and on Dow Jones Industrial (DJI) index in our analysis. We selected SP500 and DJI due to their importance as indicators of the overall market performance and, therefore, they consist the most important benchmarks of active portfolio fund managers when they evaluate their performance compared to the overall stock market. Furthermore, they have little systematic risk.

Since we wanted a real data testing and analysis procedure, we rejected the idea of constructing portfolios and then having examined the performance of each measure. By doing so, we eliminate the existence of personal prejudice in our analysis. Instead we select to base our investigation on the said indices, which are in fact portfolios. They may be considered as conservative portfolios but they are the most representative, and since we are aiming to impartial conclusions the choice of both SP500 and DJI was inevitable. Furthermore, their use has three supplementary benefits. First, they are easily identified. Second, the stocks they are consisted of represent, almost certainly, the portfolio manager's top selection, and third, by order of weight, they have the largest impact on portfolio's performance.

We design our experiment as follows. First, we travel back in time and we assumed we are living in January 1997. Therefore, the most resent observation we have on SP500 and DJI are those of December 1996. We further assumed that we are interested in forecasting the values of these two indices for an investment horizon of ten years, i.e. December 2006. The available data are the annual closing prices from 1950 to 1996 and, since we do not wish to loose valuable information, we compute the various mean values using all existed data.

All the above-mentioned mean values, i.e. arithmetic, geometric, logarithmic, identric, Blume's and Cooper's suggestions were computed and terminal values of December 2006 were estimated for both SP500 and DJI. So for each index we obtained six different values based on each one of the estimated means. The corresponding estimated values together with the actual ones are tabulated in Table I and are shown diagrammatically in Figures 1 and 2. Arithmetic-1 is the estimated terminal value based on the arithmetic mean of annual data.

As we said earlier, all these estimates are biased. So, at this moment, we follow our suggestion for unbiased estimation. We select only five observations and we ignore the rest. Namely, we select 1956, 1966, 1976, 1986 and 1996, i.e. one for every ten-year period since our investment horizon has ten years length. We, then, compute the arithmetic mean of these five observations, denoted by Arithmetic-10, and we apply it in order to estimate the value of each index for the next decade. Since our estimation is based on the arithmetic mean, we know that estimates of only one time-period ahead will be unbiased. So, our estimate for 2006 is indeed unbiased.

As we can see in Table I and in Figures 1 and 2 as well, the results are identical for SP500 and DJI. In both cases, all estimates overestimate the true value, with Arithmetic-10 always being the closest to the actual. The geometric mean is second best, followed by

Table I.
Estimated values for a ten-year investment horizon

	S&P 500	DJ Industrial
Actual	1,418.30	12,463.15
Arithmetic-10	1,550.24	13,122.20
Geometric	1,617.18	13,242.35
Logarithmic	1,708.83	14,019.83
Identric	1,708.92	14,020.59
Blume	1,768.65	14,528.51
Arithmetic-1	1,805.49	14,841.36
Cooper	1,853.35	15,248.58

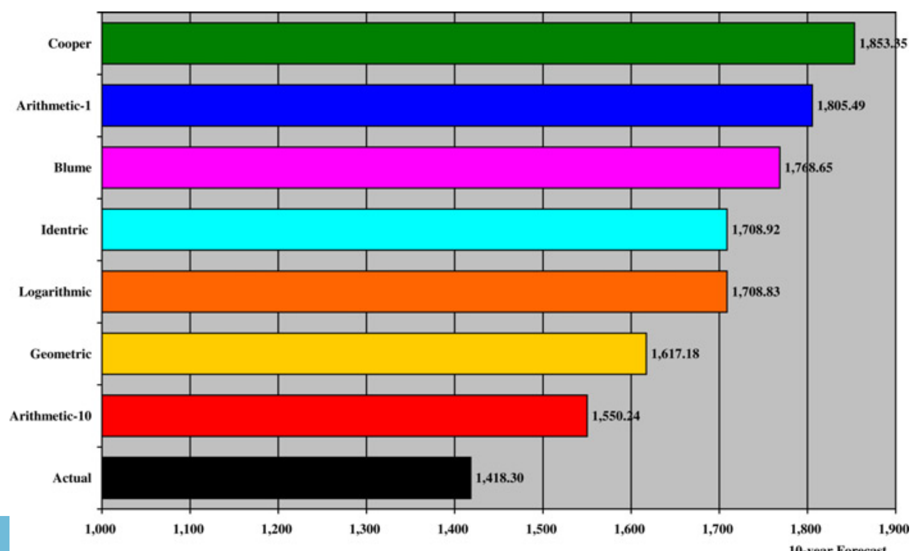

Figure 1.
S&P 500 (ten year forecast)

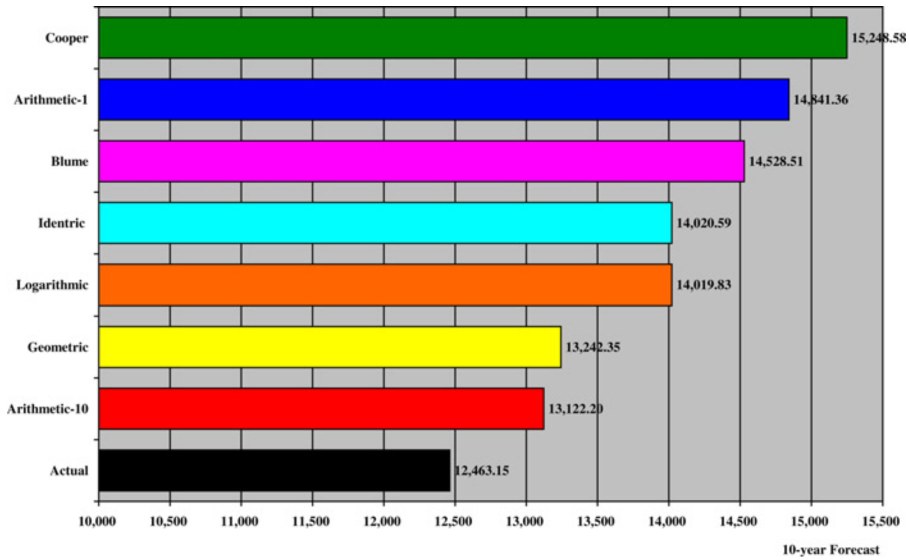

Figure 2.
Dow Jones Industrial (ten year forecast)

two equal the logarithmic and the identric means. The three furthest to the actual value are Blume's, Arithmetic-1 and Cooper's with Cooper's suggestion always being the worst.

Given that our ten-year investment horizon contains nine more years, we decided to investigate the performance of our suggestion within the entire interval of the ten-year investment horizon. We, therefore, computed in the exactly similar way Arithmetic-2, Arithmetic-3, Arithmetic-4, Arithmetic-5, Arithmetic-6, Arithmetic-7, Arithmetic-8 and Arithmetic-9 and compared them with all other means for the whole ten-year period, from 1997 to 2006.

The method we used for our investigation was to employ loss functions or error statistics. The most commonly used (see Balaban *et al.*, 2006; Brailsford and Faff, 1996) ones are the ME, the MAE, the RMSE, and the MAPE, which are defined as:

$$ME = \frac{1}{N}\sum_{j=T+1}^{T+N}\left(\hat{r}_j - r_j\right),$$

$$MAE = \frac{1}{N}\sum_{j=T+1}^{T+N}\left|\hat{r}_j - r_j\right|,$$

$$RMSE = \sqrt{\frac{1}{N}\sum_{j=T+1}^{T+N}\left(\hat{r}_j - r_j\right)^2},$$

$$MAPE = \frac{1}{N}\sum_{j=T+1}^{T+N}\left|\frac{\hat{r}_j - r_j}{r_j}\right|,$$

where r_j is the actual value of index of return at time t, $\hat{r}_j$ is the corresponding predicted value, T is the number of time-periods on which we based our prediction, and N is the investment horizon measured with the same time unit as T.

In Tables II and III, all computed values for each error statistic for SP500 and DJI respectively, are shown. For SP500, Arithmetic-10 is the unquestionable winner. In three out of four produced the best results and in only one case it was second best to logarithmic. For DJI, Arithmetic-10 was the best performer with respect to MAPE, and the second best to all other error statistics.

The estimated values are shown in Figures 3 and 4. Since the logarithmic and the identric means produced almost identical results, we are graphing only the logarithmic estimates.

For the first half of the investment horizon the performance of Arithmetic-10 is very poor with Arithmetic-1 being the best. At the second half, the picture is transformed into the exact opposite until we reach to the terminal year where Arithmetic-10 is the best. The same applies to both indices. Without loss of generality, we may deduct that our suggested method becomes more attractive as the length of the investment horizon grows.

Table II.
Error statistics for S&P 500

	Arithmetic-10	Arithmetic-1	Geometric	Logarithmic	Identric	Blume	Cooper
M.E.	−27.72	48.37	−33.86	6.45	6.48	36.80	64.23
M.A.E.	244.35	302.09	251.63	275.37	275.39	291.47	316.06
R.M.S.E.	279.37	329.73	281.59	301.85	301.87	318.93	344.54
M.A.P.E.	0.2255	0.2517	0.2354	0.2431	0.2431	0.2466	0.2578

Table III.
Error statistics for DJ Industrial

	Arithmetic-10	Arithmetic-1	Geometric	Logarithmic	Identric	Blume	Cooper
M.E.	−174.14	278.34	−430.42	−83.22	−82.88	179.12	414.82
M.A.E.	1,475.84	1,819.54	1,405.33	1,606.71	1,606.90	1,738.14	1,924.78
R.M.S.E.	1,742.30	2,019.12	1,667.59	1,796.08	1,796.25	1,926.89	2,147.67
M.A.P.E.	0.1569	0.1773	0.1579	0.1680	0.1680	0.1726	0.1828

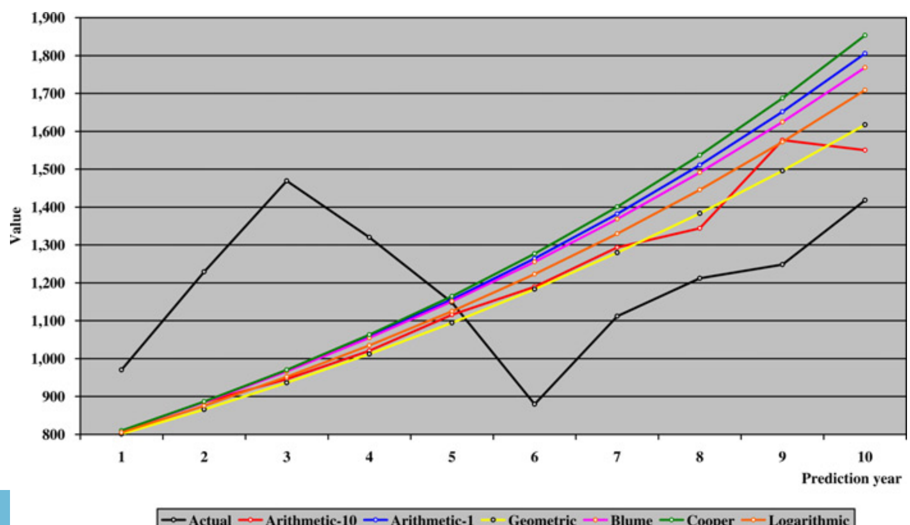

Figure 3.
S&P 500
(annual forecasts)

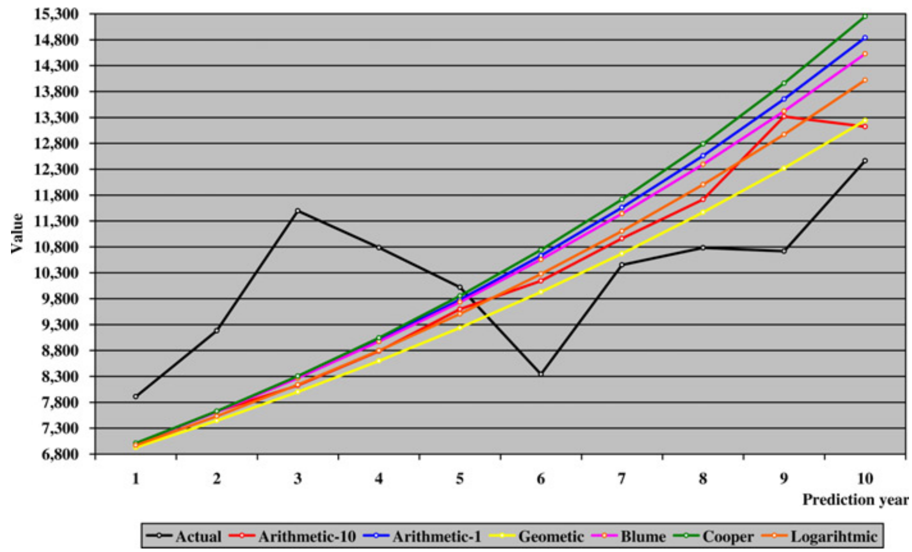

Figure 4.
Dow Jones Industrial (annual forecasts)

6. Conclusions

Whenever, in portfolio management, we are interested in producing unbiased estimates for long-term investment horizons, we are confronted with a dead end situation. All suggested estimates based on any averaging method produce biased results. Only those estimates, that are based on the arithmetic mean and estimate for only one time-period ahead, are unbiased. Such a situation is, unfortunately, inapplicable to long investment horizons. We suggest to ignore some of the available data in order to transform the long investment horizon estimation into an one time-period estimation so the results will be unbiased. By doing so, we loose something – we gain something. Losing information is the price we have to pay for having unbiased results.

From the practitioner's point of view, ignoring some information is not very important as long as the outcome is an easily applicable working tool that produces useful and sensible results. For academic scholars, on the other hand, it is a situation that needs to be examined even further in order to realize its true consequences.

Our empirical analysis, however, for SP500 and DJI showed us that our suggested method always produced the best results. The longer the investment horizon, the better, and this was happening in both SP500 and DJI. For us, these empirical results were very encouraging for using our suggested method for producing unbiased estimates for long investment horizons. Although our analysis has been conducted entirely on a small selection of widely accepted indices, we may state, without loss of generality, that our results apply to all other similar cases.

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